

# Potpourri, 4

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## Contents

<b>1</b>	<b>Random polynomials</b>	<b>1</b>
<b>2</b>	<b>Vector-valued functions</b>	<b>3</b>
<b>3</b>	<b>Polynomials, continued</b>	<b>7</b>
<b>4</b>	<b>An integral operator</b>	<b>8</b>

## 1 Random polynomials

Fix a positive integer  $n$ , and let us write  $\mathcal{B}_{n+1}$  for the set of binary strings of length  $n + 1$  written multiplicatively. To be more precise, an element  $s$  of  $\mathcal{B}_{n+1}$  is an  $(n + 1)$ -tuple  $s = (s_0, \dots, s_n)$  such that each  $s_j$  is either 1 or  $-1$ . Put  $r_j(s) = s_j$  for each  $j$ , the  $j$ th Rademacher function on  $\mathcal{B}_{n+1}$ . Let  $a_0, \dots, a_n$  be complex numbers, so that we get a polynomial

$$(1.1) \quad p(z) = \sum_{j=0}^n a_j z^j,$$

where  $z^j$  is interpreted as being equal to 1 when  $j = 0$ , as usual. For each  $s \in \mathcal{B}_{n+1}$  consider the polynomial

$$(1.2) \quad p_s(z) = \sum_{j=0}^n a_j r_j(s) z^j.$$

Let us write  $\mathbf{T}$  for the unit circle in the complex plane, which is to say the set of complex numbers  $z$  with  $|z| = 1$ . For each positive integer  $j$  we have that

$$(1.3) \quad \int_{\mathbf{T}} z^j |dz| = 0,$$

and hence also

$$(1.4) \quad \int_{\mathbf{T}} \bar{z}^j |dz| = 0,$$

where  $\bar{z}$  denotes the complex conjugate of  $z$ . Of course

$$(1.5) \quad \frac{1}{2\pi} \int_{\mathbf{T}} |dz| = 1.$$

As usual, this leads to

$$(1.6) \quad \frac{1}{2\pi} \int_{\mathbf{T}} |p(z)|^2 |dz| = \sum_{j=0}^n |a_j|^2.$$

Similarly,

$$(1.7) \quad \frac{1}{2\pi} \int_{\mathbf{T}} |p_s(z)|^2 = \sum_{j=0}^n |a_j|^2$$

for all  $s \in \mathcal{B}_{n+1}$ .

Now consider

$$(1.8) \quad 2^{-n-1} \sum_{s \in \mathcal{B}_{n+1}} \frac{1}{2\pi} \int_{\mathbf{T}} |p_s(z)|^4 |dz|.$$

By interchanging the order of summation and integration we can rewrite this as

$$(1.9) \quad \frac{1}{2\pi} \int_{\mathbf{T}} 2^{-n-1} \sum_{s \in \mathcal{B}_{n+1}} |p_s(z)|^4 |dz|.$$

There is a positive real number  $C$  such that

$$(1.10) \quad 2^{-n-1} \sum_{s \in \mathcal{B}_{n+1}} \left| \sum_{j=0}^n b_j r_j(s) \right|^4 \leq C \left( \sum_{j=0}^n |b_j|^2 \right)^2$$

for any complex numbers  $b_0, \dots, b_n$ , as one can show by expanding

$$(1.11) \quad \left| \sum_{j=0}^n b_j r_j(s) \right|^4$$

as a quadruple sum and then summing over  $s$  first. It follows that

$$(1.12) \quad 2^{-n-1} \sum_{s \in \mathcal{B}_{n+1}} \frac{1}{2\pi} \int_{\mathbf{T}} |p_s(z)|^4 |dz| \leq C \left( \sum_{j=0}^n |a_j|^2 \right)^2.$$

More generally, suppose that  $m$  is a positive integer, and consider

$$(1.13) \quad 2^{-n-1} \sum_{s \in \mathcal{B}_{n+1}} \frac{1}{2\pi} \int_{\mathbf{T}} |p_s(z)|^{2m} |dz|.$$

As before there is a positive real number  $C(m)$  such that

$$(1.14) \quad 2^{-n-1} \sum_{s \in \mathcal{B}_{n+1}} \left| \sum_{j=0}^n b_j r_j(s) \right|^{2m} \leq C(m) \left( \sum_{j=0}^n |b_j|^2 \right)^m$$

for any complex numbers  $b_0, \dots, b_n$ . By interchanging the sum with the integral and then applying this fact one obtains that

$$(1.15) \quad 2^{-n-1} \sum_{s \in \mathcal{B}_{n+1}} \frac{1}{2\pi} \int_{\mathbf{T}} |p_s(z)|^4 |dz| \leq C(m) \left( \sum_{j=0}^n |a_j|^2 \right)^m.$$

## 2 Vector-valued functions

Let  $E$  be a nonempty finite set, and let  $V$  be a real or complex vector space. The vector space of functions on  $E$  with values in  $V$  will be denoted  $\mathcal{F}(E, V)$ .

In particular,  $\mathcal{F}(E, \mathbf{R})$ ,  $\mathcal{F}(E, \mathbf{C})$  denote the spaces of real and complex-valued functions on  $E$ , respectively. These are commutative algebras with respect to pointwise multiplication of functions.

Actually, we can think of  $\mathcal{F}(E, V)$  as a module over  $\mathcal{F}(E, \mathbf{R})$  or  $\mathcal{F}(E, \mathbf{C})$ , according to whether  $V$  is a real or complex vector space. In other words, a function on  $E$  with values in  $V$  can be multiplied pointwise by a scalar-valued function in a way that is compatible with addition and scalar multiplication. For that matter, scalar multiplication amounts to the same thing as multiplication by constant scalar-valued functions.

We can also think of  $\mathcal{F}(E, V)$  as a module over the algebra of linear transformations on  $V$ , where a linear transformation on  $V$  induces a linear transformation on  $V$ -valued functions on  $E$  by acting on the values pointwise. The actions on  $\mathcal{F}(E, V)$  by pointwise multiplication by scalar-valued

functions on  $E$ , and by pointwise action by linear transformations on  $V$ , obviously commute with each other.

Suppose that  $V$  is equipped with a norm  $\|v\|_V$ . Thus  $\|v\|_V$  is a nonnegative real number for each  $v \in V$  which is equal to 0 if and only if  $v = 0$ ,

$$(2.1) \quad \|\alpha v\|_V = |\alpha| \|v\|_V$$

for all real or complex numbers  $\alpha$ , as appropriate, and all  $v \in V$ , and

$$(2.2) \quad \|v + w\|_V \leq \|v\|_V + \|w\|_V$$

for all  $v, w \in V$ . This choice of norm on  $V$  leads to a metric  $\|v - w\|_V$  on  $V$ , as usual.

Let  $p$  be given,  $1 \leq p \leq \infty$ . If  $f(x)$  is a real or complex-valued function on  $E$ , put

$$(2.3) \quad \|f\|_p = \left( \sum_{x \in E} |f(x)|^p \right)^{1/p}$$

when  $p < \infty$  and

$$(2.4) \quad \|f\|_\infty = \max\{|f(x)| : x \in E\}.$$

As is well-known, these define norms on the vector spaces of real and complex-valued functions on  $E$ . Moreover, if  $1 \leq p \leq q \leq \infty$  and  $f$  is a real or complex-valued function on  $E$ , then

$$(2.5) \quad \|f\|_q \leq \|f\|_p \leq |E|^{(1/p)-(1/q)} \|f\|_q,$$

where  $|E|$  denotes the number of elements in  $E$ .

If  $f$  is a  $V$ -valued function on  $E$ , put

$$(2.6) \quad \|f\|_{p,V} = \left( \sum_{x \in E} \|f(x)\|_V^p \right)^{1/p}$$

when  $1 \leq p < \infty$  and

$$(2.7) \quad \|f\|_{\infty,V} = \max\{\|f(x)\|_V : x \in E\}.$$

These define norms on  $\mathcal{F}(E, V)$ . Once again we have that

$$(2.8) \quad \|f\|_{q,V} \leq \|f\|_{p,V} \leq |E|^{(1/p)-(1/q)} \|f\|_{q,V}$$

when  $1 \leq p \leq q \leq \infty$  and  $f \in \mathcal{F}(E, V)$ .

Let  $V$  be a finite-dimensional real or complex vector space. By a *linear functional* on  $V$  we mean a linear mapping from  $V$  into the real or complex numbers, as appropriate. One can add linear functionals on  $V$  and multiply them by scalars in the usual manner, so that the space  $V^*$  of linear functionals on  $V$  is also a real or complex vector space, as appropriate, called the dual of  $V$ .

If  $v_1, \dots, v_n$  is a basis for  $V$ , so that every element of  $V$  can be expressed in a unique manner as a linear combination of the  $v_j$ 's, then a linear functional  $\lambda$  on  $V$  is determined uniquely by the  $n$  scalars  $\lambda(v_1), \dots, \lambda(v_n)$ , and for each choice of  $n$  scalars there is a linear functional on  $V$  whose values on the basis vectors are those scalars. In particular,  $V^*$  is also finite-dimensional and has the same dimension as  $V$ .

Now suppose that  $V$  is equipped with a norm  $\|v\|_V$ . If  $\lambda$  is a linear functional on  $V$ , then there is a nonnegative real number  $k$  such that

$$(2.9) \quad |\lambda(v)| \leq k \|v\|_V$$

for all  $v \in V$ . Indeed,  $V$  is isomorphic to  $\mathbf{R}^n$  or  $\mathbf{C}^n$  as a vector space, where  $n$  is the dimension of  $V$ , and it is well-known that any norm on  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  determines the same topology as the standard Euclidean norm.

As a result,

$$(2.10) \quad \|\lambda\|_{V^*} = \sup\{|\lambda(v)| : v \in V, \|v\|_V \leq 1\}$$

is finite, and it is the same as the smallest nonnegative real number which can be used as  $k$  in the previous inequality. This defines a norm on the dual space  $V^*$ , which is the dual norm associated to  $\|\cdot\|_V$ .

Note that

$$(2.11) \quad \|v\|_V = \sup\{|\lambda(v)| : \lambda \in V^*, \|\lambda\|_{V^*} \leq 1\}.$$

More precisely,  $\|v\|_V$  is greater than or equal to  $|\lambda(v)|$  for all linear functionals  $\lambda$  on  $V$  with dual norm less than or equal to 1 by definition, and there is such a linear functional with  $\lambda(v)$  equal to the norm of  $v$  by famous duality results.

As a basic example, let  $E$  be a nonempty finite set, and let us consider the vector space of real or complex-valued functions on  $E$ . If  $h$  is a real or complex-valued function on  $E$ , then we get a linear functional  $\lambda_h$  on the

vector space of real or complex-valued functions on  $E$ , as appropriate, by setting

$$(2.12) \quad \lambda_h(f) = \sum_{x \in E} f(x) h(x)$$

for  $f$  in the vector space. Every linear functional on the vector space of functions on  $E$  arises in this manner.

If  $1 \leq p, q \leq \infty$  are conjugate exponents, in the sense that  $(1/p) + (1/q) = 1$ , then Hölder's inequality implies that

$$(2.13) \quad |\lambda_h(f)| \leq \|f\|_p \|h\|_q$$

for all functions  $f, h$  on  $E$ , where  $\|f\|_p, \|h\|_q$  are as in the previous section. With respect to the norm  $\|f\|_p$  on functions on  $E$ , the dual norm of the linear functional  $\lambda_h$  is therefore less than or equal to  $\|h\|_q$ , and in fact one can check that it is equal to  $\|h\|_q$ .

More generally we can consider  $V$ -valued functions on  $E$ . If  $h$  is a function on  $E$  with values in  $V^*$ , then we can define a linear functional  $\lambda_h$  on the vector space of  $V$ -valued functions on  $E$  by saying that  $\lambda_h(f)$  is obtained by applying  $h(x)$  as a linear functional on  $V$  to  $f(x)$  as an element of  $V$  for each  $x \in E$ , and then summing over  $x \in E$ . One can check that every linear functional on  $\mathcal{F}(E, V)$  occurs in this way, so that the dual of  $\mathcal{F}(E, V)$  can be identified with  $\mathcal{F}(E, V^*)$ .

Using Hölder's inequality it is easy to check that  $|\lambda_h(f)|$  is less than or equal to the product of  $\|f\|_{p,V}$  and  $\|h\|_{q,V^*}$  for all  $f \in \mathcal{F}(E, V)$  and  $h \in \mathcal{F}(E, V^*)$  when  $1 \leq p, q \leq \infty$  are conjugate exponents. Furthermore, the dual norm of  $\lambda_h$  with respect to the norm  $\|f\|_{p,V}$  on  $\mathcal{F}(E, V)$  is exactly equal to  $\|h\|_{q,V^*}$ .

Let  $V$  be a finite-dimensional real or complex vector space, and let  $E$  be a nonempty finite set. If  $f$  is a  $V$ -valued function on  $E$ , then put

$$(2.14) \quad \|f\|_{p,\nu} = \sup \left\{ \left( \sum_{x \in E} |\lambda(f(x))|^p \right)^{1/p} : \lambda \in V^*, \|\lambda\|_{V^*} \leq 1 \right\}$$

when  $1 \leq p < \infty$  and

$$(2.15) \quad \|f\|_{\infty,\nu} = \sup \{ \max\{|\lambda(f(x))| : x \in E\} : \lambda \in V^*, \|\lambda\|_{V^*} \leq 1 \}.$$

One can check that these define norms on  $\mathcal{F}(E, V)$ .

For each  $v \in V$  we have that  $\|v\|_V$  is equal to the maximum of  $|\lambda(v)|$  over  $\lambda \in V^*$  with  $\|\lambda\|_{V^*} \leq 1$ , as discussed in the previous section. It is easy to see that

$$(2.16) \quad \|f\|_{p,\nu} \leq \|f\|_{p,V}$$

when  $1 \leq p < \infty$ , and that

$$(2.17) \quad \|f\|_{\infty,\nu} = \|f\|_{\infty,V}.$$

All of these norms reduce to  $\|\cdot\|_V$  when  $E$  has just one element.

We can also express  $\|f\|_{p,\nu}$  as

$$(2.18) \quad \|f\|_{p,\nu} = \sup \left\{ \left| \sum_{x \in E} h(x) \lambda(f(x)) \right| : \|h\|_q \leq 1, \lambda \in V^*, \|\lambda\|_{V^*} \leq 1 \right\}.$$

Here  $q$  is the conjugate exponent associated to  $p$ , so that  $1 \leq q \leq \infty$  and  $(1/p) + (1/q) = 1$ , and  $h$  is a real or complex-valued function on  $E$ , as appropriate. Of course  $\sum_{x \in E} h(x) \lambda(f(x))$  is the same as  $\lambda$  applied to  $\sum_{x \in E} h(x) f(x)$ , and it follows that  $\|f\|_{p,\nu}$  is equal to the supremum of the  $V$ -norm of  $\sum_{x \in E} h(x) f(x)$  over all scalar-valued functions  $h$  on  $E$  with  $\|h\|_q \leq 1$ .

### 3 Polynomials, continued

Of course a polynomial on the complex plane has the form

$$(3.1) \quad p(z) = a_n z^n + \cdots + a_1 z + a_0,$$

where  $a_0, \dots, a_n$  are complex numbers.

Alternatively one might consider polynomials in the real and imaginary parts of  $z$ , which is equivalent to polynomials in  $z$  and  $\bar{z}$ . A special case of this is given by linear combinations of powers of  $z$  and of powers of  $\bar{z}$ , including constant terms, without products of positive powers of  $z$  and  $\bar{z}$ . One might also consider linear combinations of powers of  $z$  and of  $z^{-1}$ , including constant terms. These classes all define the same functions on the unit circle, where  $|z|^2 = z\bar{z} = 1$ . Let us note that if  $f(z)$  is one of these more general kinds of polynomials, then there is a complex polynomial  $p(z)$  as in the preceding paragraph such that  $|f(z)| = |p(z)|$  for all  $z \in \mathbf{C}$  with  $|z| = 1$ .

Let us restrict our attention to complex polynomials as in (3.1). If  $p(z)$  is as in (3.1), put

$$(3.2) \quad \|p\| = \sup\{|p(z)| : z \in \mathbf{C}, |z| = 1\}$$

and

$$(3.3) \quad \|p\|_1 = \sum_{j=0}^n |a_j|.$$

These define norms on the complex vector space of polynomials, and we have that

$$(3.4) \quad \|p\| \leq \|p\|_1.$$

If  $p, q$  are complex polynomials, then

$$(3.5) \quad \|p q\| \leq \|p\| \|q\|$$

and

$$(3.6) \quad \|p q\|_1 \leq \|p\|_1 \|q\|_1.$$

If  $p(z)$  is as in (3.1), then

$$(3.7) \quad \sum_{j=0}^n |a_j|^2 = \frac{1}{2\pi} \int_{\mathbf{T}} |p(z)|^2 |dz| \leq \|p\|^2,$$

where  $\mathbf{T}$  denotes the unit circle in  $\mathbf{C}$ . One can use this to estimate  $\|p\|_1$  in terms of  $\|p\|$  and  $\|p'\|$ , where

$$(3.8) \quad p'(z) = n a_n z^{n-1} + \cdots + a_1$$

is the derivative of  $p(z)$ . As a result one can show that

$$(3.9) \quad \|p\| = \lim_{l \rightarrow \infty} \|p^l\|_1^{1/l}.$$

## 4 An integral operator

Let  $V$  denote the vector space of continuous complex-valued functions on the unit interval  $[0, 1]$  in the real line. If  $f \in V$ , then we put

$$(4.1) \quad \|f\| = \sup\{|f(x)| : 0 \leq x \leq 1\},$$

which is the usual supremum norm of  $f$ . Define a linear operator  $T$  on  $V$  by

$$(4.2) \quad T(f)(x) = \int_0^x f(s) ds.$$



If  $f$  happens to be real-valued, then  $T(f)$  is real-valued, and if  $f(x) \geq 0$  for all  $x \in [0, 1]$  too, then  $T(f)(x) \geq 0$  for all  $x \in [0, 1]$  as well. Notice that

$$(4.3) \quad \|T(f)\| \leq \|f\|$$

for all  $f \in V$ , and that equality holds when  $f$  is the constant function equal to 1.

For each positive integer  $n$  let  $T^n$  denote the  $n$ -fold composition of  $T$  on  $V$ . Equivalently, this is equal to  $T$  when  $n = 1$ , and in general  $T^{n+1}(f) = T(T^n(f))$ . One can express  $T^n(f)$  explicitly as an  $n$ -fold integral of  $f$ , and observe that  $T^n(f)$  is real-valued when  $f$  is real-valued and nonnegative when  $f$  is nonnegative. If  $f$  is the constant function equal to 1, then  $T^n(f)(1) = 1/n!$ . Indeed,  $T^n(f)(1)$  is equal to the volume of the points  $x = (x_1, \dots, x_n)$  in  $\mathbf{R}^n$  such that  $0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1$ . Using this one can check that if  $f$  is the constant function equal to 1, then  $\|T^n(f)\| = 1/n!$ . For any function  $f \in V$  we have that  $\|T^n(f)\| \leq \frac{1}{n!}\|f\|$ .

For each  $f \in V$  we also have that

$$(4.4) \quad \|T(f)\| \leq \int_0^1 |f(y)| dy.$$

As a result, if  $f_1, \dots, f_l$  are elements of  $V$ , then

$$(4.5) \quad \sum_{j=1}^l \|T(f_j)\| \leq \left\| \sum_{j=1}^l |f_j| \right\|.$$

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